Suggested Solutions to: Resit Exam, Fall 2020 Contract Theory February 22, 2021

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Question 1: Model hazard with additional information about the agent's behavior

(a) Write down expressions for the objective function and the constraints in the principal's optimization problem. One of the constraints should be an individual rationality constraint, which we refer to as the "IR constraint".

To induce the outcome $e = 1$, the principal should choose t_1, t_2, t_3, and t_4 so as to maximize

$$V = \pi_1 \left(\overline{S} - t_4\right) + (1 - \pi_1)\left(\underline{S} - t_3\right), \tag{1}$$

subject to the following constraints:

$$\pi_1 t_4 + (1 - \pi_1) t_3 - \psi \geq 0, \tag{IR}$$

$$\pi_1 t_4 + (1 - \pi_1) t_3 - \psi \geq (1 - \gamma)\left[\pi_0 t_4 + (1 - \pi_0) t_3\right] + \gamma \left[\pi_0 t_2 + (1 - \pi_0) t_1\right], \tag{IC}$$

$$t_1 \geq 0, \quad t_2 \geq 0, \quad t_3 \geq 0, \quad t_4 \geq 0. \tag{LL}$$

The objective function V is the principal's expected net surplus. The constraint IR is an individual rationality (or participation) constraint; it ensures that the agent's payoff is at least equal to the outside option payoff (which is assumed to be zero). The constraint IC is an incentive compatibility constraint, and it ensures that the agent indeed makes an effort ($e = 1$). The constraints LL are the limited liability constraints that are specified in the question.

(b) Show that the IR constraint is implied by other constraints in the problem.

We can write:

$$\pi_1 t_4 + (1 - \pi_1) t_3 - \psi \geq (1 - \gamma)\left[\pi_0 t_4 + (1 - \pi_0) t_3\right] + \gamma \left[\pi_0 t_2 + (1 - \pi_0) t_1\right] \geq 0,$$

where the first inequality holds due to IC and the second one holds due to LL and the assumptions $\gamma \in (0, 1)$ and $\pi_0 \in (0, 1)$. That is, the individual rationality constraint IR is implied by IC and LL (in conjunction with the model assumptions about γ and π_0).

(c) What are the optimal choices of t_1, t_2, t_3, and t_4?

To solve for the optimal transfers, note the following things:

- By the argument in (b) above, we can ignore the IR constraint.
- The IC constraint must bind at the optimum. To prove this, suppose the opposite—the IC constraint is lax at the optimum. If so, we must have either $t_3 > 0$ or $t_4 > 0$ (or both), and we could lower that transfer level while still satisfying the IC constraint and without violating any of the other constraints. Moreover, doing this would increase the objective, which contradicts the assumption that we started at an optimum.
- At the optimum, we must have $t_1 = t_2 = 0$. This is true since the objective is independent of those two transfer levels and the IC constraint is relaxed if we lower t_1 and/or t_2.

Thanks to the above insights, we problem consists of maximizing the objective in (1) w.r.t. t_3 and t_4, subject to the binding IC constraint,

$$\pi_1 t_4 + (1-\pi_1)\, t_3 - \psi = (1-\gamma)\,[\pi_0 t_4 + (1-\pi_0)\, t_3] \Leftrightarrow t_4 = \frac{\psi - [(1-\pi_1) - (1-\gamma)\,(1-\pi_0)]\, t_3}{\pi_1 - (1-\gamma)\pi_0}. \quad (2)$$

Thus, the optimization problem is linear in the two choice variables. The optimum must therefore be such that either t_3 or t_4 equals zero. If the right-hand side of the last equality in (2) is increasing in t_3 (i.e., if $1-\pi_1 \leq (1-\gamma)\,(1-\pi_0)$), then $t_4 = 0$ is impossible and thus $t_3 = 0$ at the optimum. The implied value of t_4 is obtained by setting $t_3 = 0$ in (2)—this expression is stated in the analysis below.

For the remaining parameter configurations (i.e., for $1-\pi_1 > (1-\gamma)\,(1-\pi_0)$), we can find the optimum by computing the cost of inducing $e = 1$ if setting $t_3 = 0$ and if setting $t_4 = 0$, respectively, and then compare.[1] If $t_3 = 0$, then (by (2)), we have

$$t_4 = \frac{\psi - [(1-\pi_1) - (1-\gamma)\,(1-\pi_0)]\, t_3}{\pi_1 - (1-\gamma)\pi_0}.$$

and thus the cost is

$$C_4 = \frac{\pi_1 \psi}{\pi_1 - (1-\gamma)\pi_0}.$$

If $t_4 = 0$, then (again by (2)), we have

$$t_3 = \frac{\psi}{(1-\pi_1) - (1-\gamma)\,(1-\pi_0)} = \frac{\psi}{(1-\pi_0)\,\gamma - (\pi_1 - \pi_0)}.$$

and thus the cost is

$$C_3 = \frac{(1-\pi_1)\,\psi}{(1-\pi_0)\,\gamma - (\pi_1 - \pi_0)}.$$

[1] Another possibility would be to use a graphical approach.

Comparing, we have

$$C_4 < C_3 \Leftrightarrow = \frac{\pi_1 \psi}{\pi_1 - (1-\gamma)\pi_0} < \frac{(1-\pi_1)\,\psi}{(1-\pi_0)\,\gamma - (\pi_1 - \pi_0)} \Leftrightarrow -\pi_1 < 1 - \gamma,$$

which always holds. Thus, we see that the optimum is, for all parameter values, to set $t_3 = 0$. Thus, we have shown that the optimal transfer values are given by

$$t_1 = 0, \quad t_2 = 0, \quad t_3 = 0, \quad t_4 = \frac{\psi}{\pi_1 - (1-\gamma)\pi_0}.$$

Question 2: Private information about both an exogenous effort cost and an endogenous effort choice

(a) Explain in words what each one of the four last constraints (i.e., the ones with a label starting with "FOA") says and how we can understand the two variables e_A^d and e_B^d.

The agent in the model can choose any effort level in the unit interval. This means that if the principal wants to induce the agent to choose some particular effort level e', then she must ensure that the agent does not prefer any other effort level in the unit interval. This gives rise to an infinite number of incentive compatibility constraints. To reduce the number of such constraints in the optimization problem, one can sometimes make use of the so-called first-order approach. The idea behind this approach is to replace the large set of incentive compatibility constraints with a single one, namely the first order condition associated with the agent's effort choice problem. From inspection of the problem above, we can conclude that first-order approach has been used. In particular, the four FOA-constraints are first-order conditions associated with four relevant effort choice problems for the agent. The constraints FOA-A and FOA-B define the agent's optimal choice of e after having picked the contract targeted at their particular type (A or B). These constraints are obtained by differentiating the left-hand side of IR-A and IR-B, respectively, and then setting the resulting expression equal to zero. The constraints FOA-A-d and FOA-B-d define the agent's optimal choice of e after having picked the other type's contract. In other words, these constraints define the optimal deviation efforts e_A^d and e_B^d, and these effort levels therefore appear in the right-hand sides of IC-A and IC-B, respectively.

(b) Show that the IR-A constraint is implied by model assumptions and other constraints in the problem.

To show that IR-A holds, it suffices if we can show that the right-hand side of IC-A is non-negative. This right-hand side clearly (by the model assumption that $\psi(0, \theta) = 0$) equals zero if $e^d = 0$. Moreover, we know that e^d is chosen so as to maximize the value of right-hand side of IC-A, which means that (since the value zero is obtainable) it cannot drop below zero.

(c) Denote the optimal value of e_A by e_A^{SB}. Solve as much as you need of the problem in order to argue that, if λ_A is small enough, then $e_A^{SB} < e_A^{FB}$, where e_A^{FB} is the A type's first-best effort level (i.e., e_A^{FB} is implicitly defined by $\overline{S} - \underline{S} = \psi_1\left(e_A^{FB}, \theta_A\right)$). Assume that all second-order conditions associated with the problem are satisfied.

By differentiating the Lagrangian stated in the question and then setting the resulting expression equal to zero, we obtain the following first-order condition:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial e_A} &= \nu\left[\overline{S} - \underline{S} - \psi_1\left(e_A, \theta_A\right) - e_A \psi_{11}\left(e_A, \theta_A\right)\right] + \lambda_A\left[\psi_1\left(e_A, \theta_A\right) + e_A \psi_{11}\left(e_A, \theta_A\right) - \psi_1\left(e_A, \theta_A\right)\right] \\ &\quad - \lambda_B e_B^d \psi_{11}\left(e_A, \theta_A\right) + \lambda_B\left[-\psi_1\left(e_A, \theta_A\right) + \psi_1\left(e_B^d, \theta_B\right)\right] \frac{\partial e_B^d}{\partial e_A} \\ &= \nu\left[\overline{S} - \underline{S} - \psi_1\left(e_A, \theta_A\right) - e_A \psi_{11}\left(e_A, \theta_A\right)\right] + \lambda_A e_A \psi_{11}\left(e_A, \theta_A\right) - \lambda_B e_B^d \psi_{11}\left(e_A, \theta_A \right. \\ &= 0,\end{aligned}$$

where the second equality makes use of the fact that $\psi_1\left(e_A, \theta_A\right) = \psi_1\left(e_B^d, \theta_B\right)$ (this equality follows from combining FOA-A and FOA-B-d). The above first-order condition simplifies to

$$\overline{S} - \underline{S} = \psi_1\left(e_A, \theta_A\right) + \frac{e_A \psi_{11}\left(e_A, \theta_A\right)}{\nu}\left[(1-\lambda_A) e_A + \lambda_B e_B^d\right]. \tag{3}$$

If the last term in (3) is strictly positive, then we must have $e_A^{SB} < e_A^{FB}$ (because then the "effective marginal cost" of effort exceeds $\psi_1\left(e_A, \theta_A\right)$). A sufficient condition for the last term to be strictly positive is that $\lambda_A < 1$, as that ensures that the expression is square brackets is strictly positive and we know that $e_A > 0$, $e_A \psi_{11}\left(e_A, \theta_A\right) > 0$, and $\nu > 0$. That is, if λ_A is small enough (smaller than unity) at the optimum, then, for the A type, the second-best effort is smaller than the first-best effort.